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THE SU(3) GENERALIZATION OF RACAH'S SU(2j+1) ⊃ SU(2) GROUP-SUBGROUP EMBEDDING

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Abstract

Racah showed how to embed the symmetry group, $SU(2)$ or $SO(3)$, of a physical system in the general unitary group $SU(2j+1)$, where the latter group is the most general group of linear transformations of determinant 1 that leaves invariant the inner product structure of an arbitrary state space H_j of the physical system. This state space H_j is at the same time the carrier space of an irreducible representation (irrep) $[j]$ of the symmetry group. This embedding is achieved by classifying the vector space of mappings $H_j \to H_j$ as irreducible tensor operators with respect to the underlying symmetry group. These irreducible tensors are the generators of the Lie algebra of $SU(2j+1)$. Racah's method is reviewed within the framework of unit tensor operators. The generalization of this technique to the symmetry group $U(3)$ to obtain the embedding $U(3) \subset U(n)$, where $n = \dim[m]$ is the dimension of an arbitrary irrep of $U(3)$. As in the $SU(2)$ case, the group $U(3)$ is the symmetry group of a physical system, and $U(\dim[m])$ is the most general group of linear transformations that preserves the inner product structure of an arbitrary state space $H_{[m]}$ of the system. This state space $H_{[m]}$ is at the same time the carrier space of irrep $[m]$ of the symmetry group $U(3)$. Preliminary results on the Lie algebraic vanishings of $U(3)$ Racah coefficients in consequence of the embedding $SU(3) \subset E_8 \subset SU(27)$ are given.

I. INTRODUCTION

SYMMETRY techniques have played a large role in the development of quantum physics and chemistry, from the indispensable applications of angular momentum and permutational symmetry, to applications of higher symmetries originating from the families of orthogonal, unitary, symplectic groups, etc. Marcos Moshinsky and his students have contributed to this field for over forty years, especially in exploiting boson operator methods for realizing the carrier spaces of irreducible representations (irreps) of groups of physical interest. Contributions most relevant to the subject of this paper include Refs. 1 7.

One well-known example of these techniques is Racah's embedding of $SU(2)$, resp., $SO(3)$, in the general unitary group $SU(2j+1)$:

$$SU(2) \subset SU(2j+1)\ , \ \text{resp.}, SO(3) \subset SU(2j+1)\ , \ j = 1, 3/2, \ \ldots\ . \tag{1.1}$$

Mathematically, the embeddings in Eq. (1.1) can be realized in many ways, most of them trivially. The essential point of Racah's method, however, is that the $SU(2)$ [resp., $SO(3)$] is to be identified as the *symmetry group* of a given physical system, while the group $SU(2j+1)$ has quite a different origin: It is not usually a symmetry of the physical system as a whole (global symmetry), but rather is realized as a group of unitary transformations that leave invariant a given energy eigenspace of the Hamiltonian, or a model Hamiltonian, being used to describe the system. The problem posed by the embedding (1.1), and solved by Racah, is to effect the embedding is such a way that the symmetry group structure fits into the general unitary transformations of the degenerate energy eigenspace as a subgroup of transformations. It is for this reason that the method finds many applications to physical problems, sometimes unexpected.

Racah[8,9] solved this physical embedding problem by the technique of classifying the Lie algebraic generators of the group $SU(2j+1)$ as irreducible tensor operators with respect to the transformations of the energy eigenspace corresponding to the underlying symmetry group, $SU(2)$ or $SO(3)$. In this manner, one obtains a basis of the Lie algebra of $SU(2j+1)$ in which the structure constants are given in terms of the 3-j and 6-j coefficients of the symmetry group. The applications of the Lie algebra of $SU(2j+1)$, presented in this way, to atomic and nuclear spectroscopy are well-known.

We review in Section II a slight generalization of the embedding (1.1) to the form

$$U(2) \subset U(2j+1)\ , \ j = 1, 3/2, \ \ldots\ . \tag{1.2a}$$

The methods employed point the way to the $U(3)$ generalization, which is the subject of this paper. With slight modifications, one can also obtain the embedding

$$SO(3) \subset U(2j+1)\ . \tag{1.2b}$$

The generalization to $U(3)$ is the embedding expressed by

$$U(3) \subset U(\dim[m])\ , \tag{1.3a}$$

where

$$\dim[m] = (m_{13} - m_{23} + 1)(m_{13} - m_{33} + 2)(m_{23} - m_{33} + 1)/2 \tag{1.3b}$$

denotes the dimension of an arbitrary (integral) irrep

$$[m] = [m_{13}, m_{23}, m_{33}] \tag{1.4a}$$

of $U(3)$. The $m_{i3} (i = 1, 2, 3)$ are arbitrary integers (positive, zero, or negative) satisfying

$$m_{13} \geq m_{23} \geq m_{33} \; . \tag{1.4b}$$

In analogy with Racah's approach, the $U(3)$ group in Eq. (1.3a) is to be the symmetry group of the physical system, while the group $U(n)$, with $n = \dim[m]$, is to be a group of transformations of a degenerate energy eigenspace.

The major deterrent to a straightforward generalization of the embedding (1.2a) to that of relation (1.3a) is the resolution of the multiplicity problem, which afflicts all unitary groups beyond $U(2)$. This involves additional structure, ***operator pattern labelling***, which makes the associated Wigner (3-j) and Racah (6-j) coefficient structure of $U(3)$ decidedly more intricate than that of $U(2)$. Nonetheless, these additional operator labellings and their structural relations are the essential key for developing comprehensive applications of higher unitary symmetry to problems such as that posed by the embedding (1.3a).

It is convenient to present in Section II Racah's method for $U(2)$ by using directly the concept of a $U(2)$ Wigner operator. This approach places clearly in evidence the parallel route to be followed in the $U(3)$ generalization. The underlying algebra of the $U(2)$ Wigner (unit tensor) operators is an essential ingredient of this approach.

The generalization to $U(3)$ is developed in Section III with emphasis on the role of operator patterns and the $U(3)$ Wigner operator algebra.

In Racah's embedding of $SU(2)$ [or $SO(3)$] in $U(2j+1)$, the structure constants of the Lie algebra factorize into a product of WCG and Racah coefficients. The existence of a group G such that, for example, $SO(3) \subset G \subset U(2j+1)$ can lead to the vanishing of a Racah coefficient, even though all the triangle and symmetry conditions are fulfilled. The first example was given by Racah.[9] (See Vanden Berghe *et al.*[10] and also Beyer *et al.*[11] for a summary of such Lie algebraic zeroes as of 1986.) It is, therefore, natural to ask: Can nontrivial zeroes of the $SU(3)$ 6-j coefficients arise in consequence of the existence of a group G satisfying

$$SU(3) \subset G \subset SU(\dim[m]) \; ? \tag{1.5}$$

The general method of effecting the embedding (1.5) is given in Section IV, and the question of zeroes for $G = E_8$ is addressed in Section V.

II. RESUMÉ OF RACAH'S METHOD

We summarize in subsections A-J below the vector space and algebraic structures underlying Racah's $SU(2) \subset U(2j+1)$ embedding. The emphasis is on the algebraic properties of unit tensor operators, since this makes transparent the generalization to $U(3)$.

A. Vector Space:

H_j denotes a finite-dimensional inner product space with orthonormal basis given by

$$B_j = \{ |\, jm > \,|\, m = -j, -j+1, \ \ldots \ , j \} \tag{2.1a}$$

of dimension

$$\dim H_j = \dim[j] = 2j + 1 \ . \tag{2.1b}$$

Here j may be any integer or half-integer in the set

$$j \in \{0, 1/2, 1, \ldots\} \ . \tag{2.1c}$$

B. Action of the SU(2) Lie algebra on $\mathbf{H}_j$:

The angular momentum operators $J_+ = J_1 + iJ_2, J_3, J_- = J_1 - iJ_2$, where $\mathbf{J} = (J_1, J_2, J_3)$ is the total angular momentum of the physical system in question $\big($generators of the quantal rotation group, $SU(2)\big)$ have the standard action on the space H_j:

$$J_\pm \,|\, jm > = [(j \mp m)(j \pm m + 1)]^{1/2} \,|\, jm \pm 1 > , \tag{2.2a}$$

$$J_3 \,|\, jm > = m \,|\, jm > , \tag{2.2b}$$

$$\mathbf{J}^2 \,|\, jm > = j(j+1) \,|\, jm > , \tag{2.2c}$$

where $\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2$ denotes the total (squared) angular momentum.

C. Action of SU(2) on $\mathbf{H}_j$:

The action of the group $SU(2)$ generated by the Lie algebra basis $\mathbf{J} = (J_1, J_2, J_3)$ on the basis H_j is given by

$$T_U : H_j \to H_j \ , \ \text{each}\ U \in SU(2) \ , \tag{2.3a}$$

where

$$T_U \mid jm >= \sum_{m'} D^j_{m'm}(U) \mid jm' > , \tag{2.3b}$$

in which $D^j(U)$ denotes the Wigner D-matrix (see Eq. (3.86) of Ref. 12 (Vol. 8) for their explicit definition, as used here); that is, the correspondence

$$U \to D^j(U) \text{ , each } U \in SU(2) \text{ ,} \tag{2.4}$$

is a representation of the group $SU(2)$ by unitary matrices of dimension, $\dim[j] = 2j + 1$. This unitary property is also expressed by

$$T_{U^{-1}} = T_{U^\dagger} \text{ ,} \tag{2.5}$$

where $\dagger$ denotes Hermitian conjugation of U.

D. Irreducible Unit Tensor Operator Maps $\mathbf{H}_j \to \mathbf{H}_j$:

The operator maps

$$\left\langle \begin{matrix} & 0 & \\ a & & -a \\ & \alpha & \end{matrix} \right\rangle : \; H_j \to H_j \tag{2.6a}$$

defined by

$$\left\langle \begin{matrix} & 0 & \\ a & & -a \\ & \alpha & \end{matrix} \right\rangle \mid jm > = \begin{cases} C^{jaj}_{m,\alpha,m+\alpha} \mid jm+\alpha > \text{ , } a = 0, 1, \ldots, 2j \\ 0 \text{ , } a = 2j+1, \; 2j+2, \; \ldots \text{ ,} \end{cases} \tag{2.6b}$$

with $\alpha = -a, -a+1, \ldots, a$, are a basis of all mappings of the vector space H_j into itself. The C-coefficients in the definition (2.6b) are Wigner-Clebsch-Gordon (WCG) coefficients of $SU(2)$; that is, these coefficients are the elements of the real orthogonal matrix that reduces the Kronecker product representation $D^j(U) \times D^j(U)$ into a direct sum of the irreps $D^a(U)$, which we denote symbolically by

$$[j] \times [j] = \sum_{a=0}^{2j} \oplus [a] \text{ .} \tag{2.7a}$$

There are, of course, exactly $(\dim[j])^2 = (2j+1)^2$ operators defined by Eq. (2.6b), including the identity map $\left\langle \begin{matrix} & 0 & \\ 0 & & 0 \\ & 0 & \end{matrix} \right\rangle = 1$, and in accord with Eq. (2.7a), we have

$$(2j+1)^2 = \sum_{a=0}^{2j} (2a+1) \text{ .} \tag{2.7b}$$

The WCG coefficients are related to the 3-j coefficients by

$$C^{abc}_{\alpha\beta\gamma} = (-1)^{a-b+\gamma} \begin{pmatrix} a & b & c \\ \alpha & \beta & -\gamma \end{pmatrix} \Big/ (2c+1)^{1/2} \,. \tag{2.8}$$

The operators defined by Eq. (2.6b) obey the following relations:

$$\sum_{\alpha} \left\langle \begin{array}{ccc} & 0 & \\ a & & -a \\ & \alpha & \end{array} \right\rangle \left\langle \begin{array}{ccc} & 0 & \\ b & & -b \\ & \alpha & \end{array} \right\rangle^{\dagger} = \mathbf{1} \text{ ,on } H_j \,; \tag{2.9a}$$

$$\sum_{m} < jm \left| \left\langle \begin{array}{ccc} & 0 & \\ a & & -a \\ & \alpha & \end{array} \right\rangle \left\langle \begin{array}{ccc} & 0 & \\ b & & -b \\ & \beta & \end{array} \right\rangle^{\dagger} \right| jm >= \delta_{ab}\delta_{\alpha\beta}(2j+1)/(2a+1) \,. \tag{2.9b}$$

In consequence of property (2.9a), the operators defined by Eq. (2.6b) are called ***unit tensor operators*** or ***Wigner operators***.

For each $a \in \{0, 1, \ldots, 2j\}$, the set of $2a+1$ maps (2.6a), corresponding to $\alpha = -a, -a+1, \ldots, a$, transform under the action of the unitary operator T_U defined by Eq. (2.3b) according to

$$T_U \left\langle \begin{array}{ccc} & 0 & \\ a & & -a \\ & \alpha & \end{array} \right\rangle T_{U^{-1}} = \sum_{\alpha'} D^{a}_{\alpha'\alpha}(U) \left\langle \begin{array}{ccc} & 0 & \\ a & & -a \\ & \alpha' & \end{array} \right\rangle, \text{ each } U \in SU(2) \,. \tag{2.10}$$

This ***equivariance relation*** expresses the property that the set of operators

$$\left\{ \left\langle \begin{array}{ccc} & 0 & \\ a & & -a \\ & \alpha & \end{array} \right\rangle \middle| \alpha = -a, -a+1, \ldots, a \right\} \tag{2.11}$$

is an irreducible tensor operator of $SU(2)$.

E. Algebra of Unit Tensor Operators:

The set of $(2j+1)^2$ operator maps $H_j \to H_j$ defined by Eq. (2.6b) are a basis of the vector space of all linear maps $H_j \to H_j$. The scalars of this vector space are invariant operators with respect to $SU(2)$. In addition, these unit tensor operators obey the following product law:

$$\left\langle \begin{array}{ccc} & 0 & \\ b & & -b \\ & \beta & \end{array} \right\rangle \left\langle \begin{array}{ccc} & 0 & \\ a & & -a \\ & \alpha & \end{array} \right\rangle = \sum_{c} \mathbf{W}^{abc}_{000} \, C^{abc}_{\alpha,\beta,\alpha+\beta} \left\langle \begin{array}{ccc} & 0 & \\ c & & -c \\ & \alpha+\beta & \end{array} \right\rangle, \tag{2.12}$$

where $\mathbf{W}^{abc}_{000}$ denotes a ***Racah invariant operator***. Its eigenvalue on the space H_j is given by

$$W^{abc}_{000}(j) = [(2c+1)(2j+1)]^{1/2}\, W(jajb;jc)\ ; \tag{2.13a}$$

that is,

$$\mathbf{W}^{abc}_{000} \mid jm >= W^{abc}_{000}(j) \mid jm > \ . \tag{2.13b}$$

The W-coefficient in the right-hand side of Eq. (2.13a) is a Racah coefficient, which is given in terms of the 6-j coefficient by

$$W(jajb:jc) = (-1)^{2j+a+b} \begin{Bmatrix} j & a & j \\ b & j & c \end{Bmatrix} . \tag{2.13c}$$

F. Weyl Basis of the Lie Algebra of U(2j+1):

The set of n^2 Weyl basis elements of the Lie algebra of $U(n)$ is given by

$$\{E_{ik} \mid i,k = 1,2,\ \dots\ ,n\} . \tag{2.14}$$

These generators of the general unitary group $U(n)$ (indeed, of the general linear group) obey the commutation relations

$$[E_{ik}, E_{i'k'}] = \delta_{ki'} E_{ik'} - \delta_{ik'} E_{i'k} . \tag{2.15}$$

For $n = 2j+1$, we denote the set (2.14) of generators of $U(2j+1)$ by

$$\{E_{j-m'+1,j-m+1} \mid m', m = -j, -j+1,\ \dots\ ,j\} . \tag{2.16}$$

F. Racah Basis of the Lie Algebra of U(2j+1):

The real orthogonal matrix R of dimension $(2j+1)^2$ is defined by

$$(R)_{a\alpha;m'm} = [(2a+1)/(2j+1)]^{1/2} \left\langle jm' \middle| \left\langle \begin{matrix} & 0 & \\ a & & -a \\ & \alpha & \end{matrix} \right\rangle \middle| jm \right\rangle . \tag{2.17}$$

Property (2.9b) expresses the fact that the matrix R with rows enumerated by the index pairs (a,α) and columns by the index pairs (m',m) is real orthogonal. This matrix is now used to define a new basis, ***the Racah basis***, of the Lie algebra of $U(2j+1)$:

$$E^a_\alpha = \sum_{m',m} (R)_{a\alpha;m'm} \, E_{j-m'+1,j-m+1} \ , \tag{2.18}$$

where we suppress j in the notation E^a_α for the Racah basis set:

$$\{E^a_\alpha \mid a = 0, 1, \ \ldots \ , 2j; \ \alpha = -a, \ -a+1, \ \ldots \ , a\} \ . \tag{2.19}$$

Since the matrix R is real orthogonal, relation (2.18) can be inverted to give the Weyl generators in terms of the Racah generators:

$$E_{j-m'+1,j-m+1} = \sum_{a,\alpha} (R)_{a\alpha;m'm} \, E^a_\alpha \ . \tag{2.20}$$

The embedding of the quantal rotation group $SU(2)$ in $U(2j+1)$ is now obtained in the following way: The relation between $SU(2)$ angular momentum operators (J_+, J_3, J_-) and Wigner operators is given by

$$J_+ = (2\mathbf{J}^2)^{1/2} \left\langle \begin{matrix} & 0 & \\ 1 & & -1 \\ & 1 & \end{matrix} \right\rangle , J_3 = (\mathbf{J}^2)^{1/2} \left\langle \begin{matrix} & 0 & \\ 1 & & -1 \\ & 0 & \end{matrix} \right\rangle ,$$

$$J_- = (2\mathbf{J}^2)^{1/2} \left\langle \begin{matrix} & 0 & \\ 1 & & -1 \\ & -1 & \end{matrix} \right\rangle . \tag{2.21}$$

From these relations and Eqs. (2.18), we obtain the following operator identities on H_j:

$$\begin{aligned} J_+ &= -[2(2j+1)j(j+1)/3]^{1/2} \, E^1_{+1} \ , \\ J_3 &= [(2j+1)j(j+1)/3]^{1/2} \, E^1_0 \ , \\ J_- &= [2(2j+1)j(j+1)/3]^{1/2} \, E^1_{-1} \ . \end{aligned} \tag{2.22}$$

These relations give explicitly the embedding of the physical $SU(2)$ group in $U(2j+1)$.

The commutators of the angular momentum operators (J_+, J_3, J_-) with the elements (2.18) of the Racah basis are given by

$$[J_\pm \ , E^a_\alpha] = [(a \mp \alpha)(a \pm \alpha + 1)]^{1/2} \, E^a_{\alpha\pm1} \ , \tag{2.23a}$$

$$[J_3 \ , E^a_\alpha] = E^a_\alpha \ . \tag{2.23b}$$

These relations show that *the Racah basis of the Lie algebra of* $U(2j+1)$ *consists of irreducible tensor operators with respect to* $SU(2)$. (The results, Eqs. (2.23), are most easily derived by using relations (2.26) below.) Relations (2.23), in turn, imply the transformation property:

$$T_U \cdot E^a_\alpha T_{U^{-1}} = \sum_{\alpha'} D^a_{\alpha'\alpha}(U) \, E^a_{\alpha'}, \text{ each } U \in SU(2) \ . \tag{2.24}$$

G. Structure Constants in the Racah Basis:

If $X = (X_{ik})$ and $Y = (Y_{ik})$ are arbitrary ma rices of dimension n (over the complex numbers), we define the (extended) elements of the Lie algebra of $U(n)$ by

$$L_X = \sum_{i,k=1}^{n} X_{ik}\, E_{ik},\; L_Y = \sum_{i,k=1}^{n} Y_{ik}\, E_{ik} \,. \tag{2.25}$$

It is then easily proved that

$$[L_X, L_Y] = L_{[X,Y]} \,. \tag{2.26}$$

Application of this relation to any pair of elements E^a_α and E^b_β in the Racah basis and use of the product law (2.12) for unit tensor operators gives

$$[E^a_\alpha, E^b_\beta] = \sum_{c,\gamma} A^{abc}_{\alpha\beta\gamma}\, E^c_\gamma \,, \tag{2.27}$$

where the *structure constants* are given by

$$A^{abc}_{\alpha\beta\gamma} = [(-1)^{a+b-c} - 1] C^{abc}_{\alpha\beta\gamma} [(2a+1)(2b+1)]^{1/2}\, W(abjj;cj) \,. \tag{2.28}$$

(We have used well-known symmetries of the WCG and Racah coefficients to bring the structure constants to the form (2.28).)

H. Action of U(2j+1) on H_j:

Let $V \in U(2j+1)$, and let the elements of V be enumerated by $(V_{m'm})$ with $m' = j, j-1, \ldots, -j$ denoting rows (read from left to right), and $m = j, j-1, \ldots, -j$ denoting columns (read from top to bottom) in the matrix V. Then the action S_V of $U(2j+1)$ of H_j is given by

$$S_V \mid jm >= \sum_{m'} V_{m'm} \mid jm' > \text{, each } V \in U(2j+1) \,. \tag{2.29}$$

In particular, since $D^j(U) \in U(2j+1)$, we find that

$$S_{D^j(U)} = T_U \,. \tag{2.30}$$

This result shows clearly that the space H_j carries the fundamental representation

$$D^{[10\cdots0]}(V) = V \tag{2.31}$$

of $U(2j+1)$ and that this representation, when restricted to the quantal rotation subgroup $SU(2)$, as embedded in $U(2j+1)$ in the Racah basis, reduces to the irreps of this $SU(2)$:

$$D^{[10\cdots 0]}\Big(D^j(U)\Big) = D^j(U), \text{ each } U \in SU(2)\ . \qquad (2.32)$$

I. Expansion to U(2) ⊂ U(2j+1):

The generators of $U(2)$ are obtained by adjoining the invariant operator

$$E_0^0 = (2j+1)^{-1/2} \sum_{i=1}^{2j+1} E_{ii} \qquad (2.33)$$

to the $SU(2)$ generators (J_+, J_3, J_-). The transition to $U(2)$ is then best made by using the full $U(2)$ Gel'fand patterns as explained in detail in Ref. 12.

J. The Modification to SO(3) ⊂ U(2j+1):

The results given in A-H above apply equally well to the group $SO(3)$ of 3×3 real, proper orthogonal matrices corresponding to the rotations of a physical system in $\mathbb{R}^3$. It is customary for this case to write $j = \ell, \mathbf{J} = \mathbf{L} = (L_1, L_2, L_3)$, and to replace $D^\ell(U)$ by $\mathcal{D}^\ell(R) = D^\ell\Big(U(R)\Big)$, where $\pm U(R) \rightarrow R$ in the two-to-one homomorphism of $SU(2)$ onto $SO(3)$.

We conclude this summary of Racah's embedding as described in A-H above by again pointing out that $SU(2)$ [resp., $SO(3)$] is the group under whose action the physical system is invariant (symmetry group). The group $U(2j+1)$ [resp., $U(2\ell+1)$] need not be a symmetry group of the full physical system, but only of special quantum states of the system. This general unitary group always has the significance of being the most general group of linear transformations of the space H_j that preserves the inner product structure of this space.

III. THE U(3) ⊂ U(dim[m]) EMBEDDING

We can now give the

$$U(3) \subset U(\dim[m]) \qquad (3.1)$$

generalization of Racah's $U(2) \subset U(2j+1)$ embedding by following, step by step, the procedures given in A-I in Section II. The group $U(3)$, or $SU(3)$, is taken as the symmetry group of a physical system, or the model of such a system. The meaning of the group $U(\dim[m])$ is that it is the most general group of linear transformations of the carrier space $H_{[m]}$ of an irrep of $U(3)$, which preserves the inner product. The basic problem is to classify the set of all maps $H_{[m]} \to H_{[m]}$ as irreducible tensor operators with respect to the underlying symmetry group $U(3)$. We proceed as in Section II.

A. Vector Space:

$H_{[m]}$ denotes a finite-dimensional inner product space with orthonormal basis given by

$$B_m = \left\{ \left| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{matrix} \right\rangle \middle| \begin{matrix} \text{this array is a} \\ \text{Gel'fand pattern} \end{matrix} \right\} . \tag{3.2}$$

We recall that a $U(3)$ Gel'fand pattern is any triangular array of integers, positive, zero, or negative, satisfying the betweenness relations:

$$\begin{aligned} m_{13} \geq m_{12} \geq m_{23} \geq m_{22} \geq m_{33} \\ m_{12} \geq m_{11} \geq m_{22} \,. \end{aligned} \tag{3.3}$$

We denote such a Gel'fand pattern by

$$\begin{pmatrix} [m] \\ m \end{pmatrix} , \; [m] = [m_{13} m_{23} m_{33}], \; m = \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} . \tag{3.4}$$

At times we use a notation less encumbered with subscripts. The dimension of the space $H_{[m]}$ is $\dim[m]$, as given by Eq. (1.3b).

B. Action of the U(3) Lie Algebra on $\mathrm{H}_{[m]}$:

We denote the set of Weyl basis elements of the Lie algebra of the symmetry group $U(3)$ of a physical system by

$$\{K_{ij} \mid i,j = 1,2,3\} . \tag{3.5}$$

These operators are, by definition, linear mappings $H_{[m]} \to H_{[m]}$ satisfying the commutation relations

$$[K_{ij}, K_{k\ell}] = \delta_{jk} K_{i\ell} - \delta_{i\ell} K_{kj} , \tag{3.6}$$

and the Hermitian conjugate relations $K_{ij}^{\dagger} = K_{ji}$. The action of these generators of $U(3)$ on $H_{[m]}$ is given by

$$K_{ij}\left|\begin{matrix}[m]\\ m\end{matrix}\right\rangle = \sum_{m'}\left\langle\begin{matrix}[m]\\ m'\end{matrix}\right|K_{ij}\left|\begin{matrix}[m]\\ m\end{matrix}\right\rangle\left|\begin{matrix}[m]\\ m'\end{matrix}\right\rangle , \tag{3.7}$$

where the matrix elements $< \ldots |K_{ij}| \ldots >$ are the standard ones.[13]

The state space of a physical system having $U(3)$ symmetry can then be decomposed as a direct sum $\sum \oplus H_{[m]}$ (including multiplicities, as required). The vector space $H_{[m]}$ used throughout this section has this significance of a subspace of states of such a physical system.

C. Action of U(3) on $h_{[m]}$.

For each $U \in U(3)$, the Lie algebra $\{K_{ij}\}$ generates a unitary transformation T_U of the space $H_{[m]}$:

$$T_U\left|\begin{matrix}[m]\\ m\end{matrix}\right\rangle = \sum_{m'} D^{[m]}_{m'm}(U)\left|\begin{matrix}[m]\\ m'\end{matrix}\right\rangle , \tag{3.8a}$$

where $D^{[m]}(U)$ denotes a unitary matrix with rows and columns enumerated by the $U(2)$ Gel'fand patterns m' and m, respectively. Here the irrep label $[m]$ occurs as the top row in both patterns:

$$\begin{pmatrix}[m]\\ m'\end{pmatrix} and \begin{pmatrix}[m]\\ m\end{pmatrix} . \tag{3.8b}$$

The unitary irreducible representation matrices

$$\{D^{m}(U) \mid U \in U(3)\} \tag{3.9}$$

are much-studied objects[5,6,14-24] and are known explicitly, but are quite complicated. The only property we need here is that the correspondence

$$U \to D^{[m]}(U) \text{ , each } U \in U(3) \tag{3.10}$$

is a representation of $U(3)$ by unitary matrices of dimension, $\dim[m]$. The unitary property is also expressed as

$$T_{U^{-1}} = T_{U^{\dagger}} . \tag{3.11}$$

D. Irreducible Unit Tensor Operator Maps $\mathbf{H}_{[m]} \rightarrow \mathbf{H}_{[m]}$:

It is well-known how to determine the irrep labels $[k] = [k_1 k_2 k_3]$ of the irreducible $U(3)$ tensor operator maps:

$$\mathbf{T}([k]) : H_{[m]} \rightarrow H_{[m]} \ . \tag{3.12}$$

The $U(3)$ irrep labels are just those occurring in the reduction of the Kronecker product

$$[\overline{m}] \times [m] = \sum \oplus [k] \ . \tag{3.13}$$

Here $[\overline{m}]$ denotes the irrep of $U(3)$ that is the complex conjugate of $[m]$:

$$[\overline{m}] = [\overline{m}_{13} \overline{m}_{23} \overline{m}_{33}] = [-m_{33}, -m_{23}, -m_{13}] \ . \tag{3.14}$$

(See, for example, Ref. 14.)

The explicit decomposition of the Kronecker product $[\overline{m}] \times [m]$ into irreps of $U(3)$ is given by

$$[\overline{m}] \times [m] = \sum_{[k]} I([\overline{m}] \times [m]; [k])[k] \ , \tag{3.15}$$

where $I([\overline{m}] \times [m]; [k])$ denotes the (intertwining) number of times irrep $[k]$ is contained in the representation $[\overline{m}] \times [m]$. (See Refs. 25-28 for a discussion of these numbers and their relation to the Littlewood-Richardson numbers.) The summation in Eq. (3.15) is over all irreps

$$[k] = [k_1 k_2 k_3] \tag{3.16a}$$

such that the k_i are integers satisfying the following relations:

$$\begin{aligned} &\text{(i)} \quad k_1 + k_2 + k_3 = 0 \ , \\ &\text{(ii)} \quad 0 \leq k_1 \leq m_{13} - m_{33} \ , \\ &\text{(iii)} \quad -(m_{13} - m_{33}) \leq k_3 \leq 0 \ . \end{aligned} \tag{3.16b}$$

The interwining number has the following explicit values:

$$I([\overline{m}] \times [m]; [k]) = k_1 - k_2 + 1 \ , \quad \text{for } k_2 \geq 0 \text{ and } k_1 \leq \min(m_{13} - m_{23}, m_{23} - m_{33}) \ ; \tag{3.17a}$$

$$I([\overline{m}] \times [m]; [k]) = \min(m_{13} - m_{23} - k_2, m_{23} - m_{33} - k_2) \ , \quad \text{for } k_2 \geq 0 \text{ and } k_1 \geq \min(m_{13} - m_{23}, m_{23} - m_{33}) \ . \tag{3.17b}$$

For $k_2 < 0$, we have the relation

$$I([\overline{m}] \times [m]\,;\,[k]) = I([\overline{m}] \times [m]\,;\,[\bar{k}])\,, \tag{3.17c}$$

so that together Eqs. (3.17a,b) and (3.17c) give explicitly all multiplicity numbers appearing in the Kronecker product reduction, Eq. (3.15).

The explicit set of unit tensor operators, which constitute a basis for all maps $H_{[m]} \to H_{[m]}$, is given by

$$\left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle = \left\langle \begin{matrix} & & 0 & & \\ & \ell & & -\ell & \\ k_1 & & k_2 & & k_3 \\ & \alpha_{12} & & \alpha_{22} & \\ & & \alpha_{11} & & \end{matrix} \right\rangle, \tag{3.18a}$$

where the irrep label $[k]$ runs over all values satisfying the conditions (i)-(iii) above. For each such $[k]$, the entry in the *operator pattern*

$$\lambda = \begin{pmatrix} & 0 & \\ \ell & & -\ell \end{pmatrix}, \tag{3.18b}$$

and the α_{ij} in the Gel'fand pattern

$$\alpha = \begin{pmatrix} \alpha_{12} & & \alpha_{22} \\ & \alpha_{11} & \end{pmatrix}, \tag{3.18c}$$

assume the following values:

$$\ell = k_2, k_2 + 1, \ \ldots\ , k_2 - 1 + I([\bar{m}] \times [m]\,;\,[k])\,; \tag{3.18d}$$

$$k_1 \geq \alpha_{12} \geq k_2 \geq \alpha_{22} \geq k_3;\ \alpha_{12} \geq \alpha_{11} \geq \alpha_{22}\,. \tag{3.18e}$$

The restriction of the ℓ-values to the domain given by Eq. (3.18d), rather than the full domain admitted by the betweenness conditions, is a consequence of the canonical splitting conditions for the $U(3)$ WCG coefficients as determined by null space (see Refs. 29-35).

That the counting is correct for the reduction given by Eqs. (3.15)-(3.18) may be verified from the equation

$$(\dim[m])^2 = \sum_{[k]} I([\bar{m}] \times [m]; [k]) \dim[k]\,. \tag{3.19}$$

where the summation is over all $[k]$ satisfying conditions (3.16b).

It is useful to give an example illustrating the above rules:

$$
\begin{aligned}
[0,-1,-2] \times [2\ 1\ 0] = [0\ 0\ 0] \oplus 2[1,0,-1] \oplus [1,1,-2] \\
\oplus [2,-1,-1] \oplus [2,0,-2]\,. \qquad (3.20)
\end{aligned}
$$

The set of patterns labelling the unit tensor operator maps $H_{[2\ 1\ 0]} \to H_{[2\ 1\ 0]}$ are

$$
\begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{pmatrix},\
\begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & & \alpha & & \end{pmatrix},\
\begin{pmatrix} & & 0 & & \\ & -1 & & -1 & \\ 1 & & 0 & & -1 \\ & & \alpha & & \end{pmatrix},
$$

$$
\begin{pmatrix} & & 0 & & \\ & 1 & & -1 & \\ 1 & & 1 & & -2 \\ & & \alpha' & & \end{pmatrix},\
\begin{pmatrix} & & 0 & & \\ & 1 & & -1 & \\ 2 & & -1 & & -1 \\ & & \alpha'' & & \end{pmatrix},\
\begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ 2 & & 0 & & -2 \\ & & \alpha''' & & \end{pmatrix}, \qquad (3.21)
$$

in which $\alpha, \alpha', \alpha'', \alpha'''$ runs over all patterns satisfying betweenness. This provides the $1 + 2(8) + 10 + 10 + 27 = 64$ mappings of the space $H_{[2\ 1\ 0]}$, with $\dim[2\ 1\ 0] = 8$, onto itself.

The action of the unit tensor operator (3.18a) on the space $H_{[m]}$ is given by

$$
\left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \left| \begin{matrix} [m] \\ m \end{matrix} \right\rangle = \sum_{m'} \left\langle \begin{matrix} [m] \\ m' \end{matrix} \right| \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \left| \begin{matrix} [m] \\ m \end{matrix} \right\rangle \left| \begin{matrix} [m] \\ m' \end{matrix} \right\rangle , \qquad (3.22)
$$

where the notation $< \cdots | < > | \cdots >$ denotes a WCG coefficient of $U(3)$. The operators defined by Eq. (3.22) obey the following relations:

$$
\sum_{\alpha} \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda' \\ [k] \\ \alpha \end{matrix} \right\rangle^{\dagger} = \delta_{\lambda\lambda'} 1 \ , \text{ on } H_{[m]} \ , \qquad (3.23a)
$$

$$
\sum_{\lambda} \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle^{\dagger} \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha' \end{matrix} \right\rangle = \delta_{\alpha\alpha'} 1 \ , \text{ on } H_{[m]} \ , \qquad (3.23b)
$$

$$\sum_{m} \left\langle \begin{matrix} [m] \\ m \end{matrix} \middle| \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda' \\ [k] \\ \alpha' \end{matrix} \right\rangle^{\dagger} \middle| \begin{matrix} [m] \\ m \end{matrix} \right\rangle = \delta_{\lambda\lambda'}\delta_{\alpha\alpha'}(\dim[m])/(\dim[k]) \,. \quad (3.23c)$$

In consequence of relation (3.23a), these operators are called unit tensor operators.

For each irrep label $[k]$, the set of $\dim[m]$ maps (3.18a) corresponding to all Gel'fand patterns α transforms under the action of the unitary operator T_U defined by Eq. (3.8) according to

$$T_U \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle T_{U^{-1}} = \sum_{\alpha'} D^{k}_{\alpha'\alpha}(U) \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha' \end{matrix} \right\rangle \,. \quad (3.24)$$

This relation equivariance expresses the property that the set of operators

$$\left\{ \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \,\middle|\, \alpha \text{ is a Gel'fand pattern} \right\} \quad (3.25)$$

is an irreducible tensor operator of $U(3)$ for each operator pattern λ.

E. Algebra of U(3) Unit Tensor Operators:

The set of $(\dim[m])^2$ operator maps $H_{[m]} \to H_{[m]}$ defined by Eqs. (3.22) is a basis of the vector space of all linear maps $H_{[m]} \to H_{[m]}$. The scalars of this vector space are invariant operators with respect to $U(3)$. In addition, these unit tensor operators obey the following ***product law***:

$$\left\langle \begin{matrix} \lambda' \\ [k'] \\ \alpha' \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle = \sum \left\{ \begin{matrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{matrix} \right\} \left\langle \begin{matrix} \lambda'' \\ [k''] \\ \alpha'' \end{matrix} \right\rangle \,, \quad (3.26)$$

where the quantity $\{ \vdots \vdots \vdots \}$ is an invariant operator; it is defined in terms of $U(3)$ WCG coefficients and $U(3)$ Racah invariant operators by

$$\left\{ \begin{matrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{matrix} \right\} = \sum_{\gamma'} \left\langle \begin{matrix} [k''] \\ \alpha'' \end{matrix} \middle| \left\langle \begin{matrix} \gamma' \\ [k'] \\ \alpha' \end{matrix} \right\rangle \middle| \begin{matrix} [k] \\ \alpha \end{matrix} \right\rangle$$
$$\times \left\{ \left(\begin{matrix} [k''] \\ \lambda'' \end{matrix} \right) \left(\begin{matrix} \gamma' \\ [k'] \\ \lambda' \end{matrix} \right) \left(\begin{matrix} [k] \\ \lambda \end{matrix} \right) \right\} \,. \quad (3.27)$$

The summation is Eq. (3.26) is over all irreps $[k'']$ contained in $[k'] \times [k]$, and also over all Gel'fand patterns α'' and operator patterns λ'', as described in Eq. (3.29b) below. In Eq. (3.27), the symbol $< \cdots | < > | \cdots >$ denotes a $U(3)$ WCG coefficient, and $\{(\cdots)(\cdots)(\cdots)\}$ denotes a $U(3)$ Racah invariant operator. The eigenvalue of this $U(3)$ Racah invariant on the space $H_{[m]}$ is denoted

$$\left\{ \begin{pmatrix} [k''] \\ \lambda'' \end{pmatrix} \begin{pmatrix} \gamma' \\ [k'] \\ \lambda' \end{pmatrix} \begin{pmatrix} [k] \\ \lambda \end{pmatrix} \right\}([m]) \,, \tag{3.28a}$$

and is defined by the action of the invariant operator on an arbitrary vector of $H_{[m]}$, which may be taken to be a basis vector:

$$\begin{aligned} &\left\{ \begin{pmatrix} [k''] \\ \lambda'' \end{pmatrix} \begin{pmatrix} \gamma' \\ [k'] \\ \lambda' \end{pmatrix} \begin{pmatrix} [k] \\ \lambda \end{pmatrix} \right\} \left| \begin{matrix} [m] \\ m \end{matrix} \right\rangle \\ &= \left\{ \begin{pmatrix} [k''] \\ \lambda'' \end{pmatrix} \begin{pmatrix} \gamma' \\ [k'] \\ \lambda' \end{pmatrix} \begin{pmatrix} [k] \\ \lambda \end{pmatrix} \right\}([m]) \left| \begin{matrix} [m] \\ m \end{matrix} \right\rangle . \end{aligned} \tag{3.28b}$$

The $U(3)$ Racah coefficient given by the notation (3.28a) is a real number of considerable complexity in its dependence on the integer entries entering its definition. For clarity, we display the symbol fully, in unabbreviated notation:

$$\left\{ \begin{pmatrix} k''_1 & & k''_2 & & k''_3 \\ & \ell'' & & -\ell'' & \\ & & 0 & & \end{pmatrix} \begin{pmatrix} & & \gamma'_{11} & & \\ & \gamma'_{12} & & \gamma'_{22} & \\ k'_1 & & k'_2 & & k'_3 \\ & \ell' & & -\ell' & \\ & & 0 & & \end{pmatrix} \begin{pmatrix} k_1 & & k_2 & & k_3 \\ & \ell & & -\ell & \\ & & 0 & & \end{pmatrix} \right\}([m_{13} m_{23} m_{33}]) \tag{3.29a}$$

By construction, this coefficient is zero unless all of the following *generalized triangle conditions* are fulfilled:

(i) $[k] \in [\bar{m}] \times [m]$ and $[k'] \in [\bar{m}] \times [m]$;

(ii) $[k''] \in [k'] \times [k]$;

(iii) $[k''] = [k] + \Delta(\gamma')$, where $\Delta(\gamma')$ denotes the *shift* of the operator pattern γ' defined by $\Delta(\gamma') = [\Delta_1(\gamma'), \Delta_2(\gamma'), \Delta_3(\gamma')]$
$= [\gamma'_{11}, \gamma'_{12} + \gamma'_{22} - \gamma'_{11}, k'_1 + k'_2 + k'_3 - \gamma'_{12} - \gamma'_{22}]$;

(iv) all patterns satisfy the betweenness relations ;

(v) $[m]$ is an arbitrary $U(3)$ irrep label . (3.29b)

Here we use $\in$ to denote, for example, that $[k]$ occurs in the Kronecker product $[\overline{m}] \times [m]$ reduction.

The above notation for a $U(3)$ Racah coefficient is explained in detail in Refs. 14,36; indeed, the notation extends to $U(n)$. It is useful to remark that only operator patterns and the irrep labels of the irrep space $H_{[m]}$ appear in this notation, the latter reflecting that the coefficients have their origin as invariant operators, and the former reflecting that operator patterns are structual elements arising in the resolution of the multiplicity of irreps in the Kronecker product for general $U(n)$.

The notation described above applies equally well to $U(2)$ Racah coefficients, which are related in the following way to the standard W or 6-j notation:

$$\left\{ \begin{pmatrix} k_1'' & & k_2'' \\ & \lambda'' & \end{pmatrix} \begin{pmatrix} & \gamma' & \\ k_1' & & k_2' \\ & \lambda' & \end{pmatrix} \begin{pmatrix} k_1 & & k_2 \\ & \lambda & \end{pmatrix} \right\}^{(m_{12}, m_{22})}$$
$$= [(2e+1)(2f+1)]^{1/2} W(abcd; ef) , \qquad (3.30a)$$

where

$$\begin{aligned}
a &= \frac{1}{2}(m_{12} - m_{22}) - [\lambda'' - \frac{1}{2}(k_1'' + k_2'')] , \\
b &= \frac{1}{2}(k_1 - k_2) , \\
c &= \frac{1}{2}(m_{12} - m_{22}) , \\
d &= \frac{1}{2}(k_1' - k_2') , \\
e &= \frac{1}{2}(m_{12} - m_{22}) - [\lambda' - \frac{1}{2}(k_1' + k_2')] , \\
f &= \frac{1}{2}(k_1'' - k_2'') ,
\end{aligned} \qquad (3.30b)$$

with $k_1'' = k_1 + \gamma', k_2'' = k_2 + (k_1' + k_2' - \gamma')$ and $\lambda'' = \lambda + \lambda'$. It is a fact that $U(n)$ Racah coefficients may always be taken (by a phase choice) to be $SU(n)$ Racah coefficients.

The $U(2)$ notation above is redundant, but nonetheless is indispensible for the $U(3)$ Racah coefficients and the general $U(n)$ case in exhibiting clearly the structure features of these coefficients. The notation shows unambiguously how Racah coefficients inherit the operator pattern from those of the Wigner coefficients, these patterns themselves being the structural elements on which is based the resolution of the multiplicity problem. This is most clearly brought out by the following identities (see Refs. 36-38) for the case at hand:

$$\left\{\begin{matrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{matrix}\right\} = \left\langle \begin{matrix} \lambda' \\ [k'] \\ \alpha' \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda'' \\ [k''] \\ \alpha'' \end{matrix} \right\rangle^{\dagger} , \tag{3.31a}$$

$$\left\{ \begin{pmatrix} [k''] \\ \lambda'' \end{pmatrix} \begin{pmatrix} \gamma' \\ [k'] \\ \lambda' \end{pmatrix} \begin{pmatrix} [k] \\ \lambda \end{pmatrix} \right\} = \sum_{\alpha,\alpha',\alpha''} \left\langle \begin{matrix} [k''] \\ \alpha'' \end{matrix} \,\middle|\, \left\langle \begin{matrix} \gamma' \\ [k'] \\ \alpha' \end{matrix} \right\rangle \,\middle|\, \begin{matrix} [k] \\ \alpha \end{matrix} \right\rangle$$

$$\times \left\langle \begin{matrix} \lambda' \\ [k'] \\ \alpha' \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \left\langle \begin{matrix} \lambda'' \\ [k''] \\ \alpha'' \end{matrix} \right\rangle^{\dagger} . \tag{3.31b}$$

It is worth remarking here that the general algebra of unit tensor operators is infinite-dimensional (countably), but since all the operators introduced in this section are maps $H_{[m]} \to H_{[m]}$, the algebra (3.26) is finite-dimensional, there being altogether $(\dim[m])^2$ elements in the algebra, which closes. Indeed, all these unit tensor operators are represented on $H_{[m]}$ by real orthogonal matrices of dimension, $\dim[m]$, a key fact that we utilize in Section G below.

F. Weyl Basis of the Lie Algebra of U(dim[m]):

The set of $(\dim[m])^2$ Weyl basis elements of $U(\dim[m])$ is given by

$$\left\{ E_{m'm} \,\middle|\, \begin{pmatrix} [m] \\ m' \end{pmatrix} \text{ and } \begin{pmatrix} [m] \\ m \end{pmatrix} \text{ are Gel'fand patterns} \right\} . \tag{3.32}$$

The subscripts m' and m are $U(2)$ Gel'fand patterns that share the same $U(3)$ irrep label $[m]$. These operators obey the commutation relations

$$[E_{m'm}, E_{m'''m''}] = \delta_{m\,m'''} E_{m'm''} - \delta_{m'm''} E_{m'''m} , \tag{3.33}$$

where for the evaluation of the Kronecker δ's two $U(2)$ Gel'fand patterns are defined to be equal when their corresponding entries are equal, and otherwise are defined to be unequal.

A bijection of the Weyl basis elements (3.32) to the standard set

$$\{E_{ij} \mid i,j = 1,2, \ldots , \dim[m]\} \tag{3.34}$$

may be obtained from the following rules: For given $U(3)$ irrep labels $[m]$, we introduce a total ordering on the set of all $U(2)$ patterns by writing

$$\begin{pmatrix} m'_{12} & & m'_{22} \\ & m'_{11} & \end{pmatrix} > \begin{pmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{pmatrix} \tag{3.35a}$$

whenever the first nonzero entry in the 3-tuple

$$(m'_{12} - m_{12}, m'_{22} - m_{22}, m'_{11} - m_{11}) \tag{3.35b}$$

is non-negative; otherwise, we write $m > m'$. The bijection is then given by

$$\begin{pmatrix} m_{13} & & m_{23} \\ & m_{13} & \end{pmatrix} \to 1, \ \ldots \ , \begin{pmatrix} m_{23} & & m_{33} \\ & m_{33} & \end{pmatrix} \to \dim[m] \ , \tag{3.36}$$

with all intermediate patterns being mapped in turn to the integer given by the rule: if $m' > m$ with $m' \to n'$ and $m \to n$, then $n' < n$. This rule accords with the one used in Eq. (2.16), where the index $j - m + 1$ is mapped to 1 for the highest weight ($m = j$) and to $2j + 1$ for the lowest weight ($m = -j$).

G. Racah Basis of the Lie Algebra of U(dim[m]):

The real orthogonal matrix R of dimension $(dim[m])^2$ is defined by

$$R\left[\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix}\begin{pmatrix} m \\ [m] \\ m' \end{pmatrix}\right] = \left[\frac{\dim[k]}{\dim[m]}\right]^{1/2} \left\langle \begin{matrix} [m] \\ m' \end{matrix} \middle| \left\langle \begin{matrix} \lambda \\ [k] \\ \alpha \end{matrix} \right\rangle \middle| \begin{matrix} [m] \\ m \end{matrix} \right\rangle . \tag{3.37a}$$

Property (3.23c) expresses the fact that the matrix R with rows and columns enumerated by the patterns

$$\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} \text{ and } \begin{pmatrix} m \\ [m] \\ m' \end{pmatrix} , \tag{3.37b}$$

respectively, is real orthogonal. This matrix is now used to define a new basis, *the Racah basis*, of the Lie algebra of $U(\dim[m])$:

$$E\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} = \sum_{m',m} R\left[\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix}\begin{pmatrix} m \\ [m] \\ m' \end{pmatrix}\right] E_{m'm} \ . \tag{3.38}$$

Since the matrix R is real orthogonal, relation (3.38) can be inverted to given the Weyl generators in terms of the Racah generators:

$$E_{m'm} = \sum_{\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix}} R\left[\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix}\begin{pmatrix} m \\ [m] \\ m' \end{pmatrix}\right] E\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} . \tag{3.39}$$

The embedding of the symmetry group $U(3)$ in the $U(\dim[m])$ is now obtained in the following way: The relation between the K_{ij} generators of $U(3)$ given by Eqs. (3.5)-(3.7) and $U(3)$ Wigner operators is given by (see Refs. 14,39,40)

$$K_{ij} = (-1)^j(\mathbf{I})^{1/2}\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & & (i,j) & & \end{array} \right\rangle , \; i \neq j , \tag{3.40a}$$

$$(K_{11} - K_{22})/\sqrt{2} = -(\mathbf{I})^{1/2}\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & 1 & & -1 & \\ & & 0 & & \end{array} \right\rangle , \tag{3.40b}$$

$$(K_{11} + K_{22} - 2K_{33})/\sqrt{6} = (\mathbf{I})^{1/2}\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle , \tag{3.40c}$$

$$K_{11} + K_{22} + K_{33} = \mathbf{I}_1 \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle . \tag{3.40d}$$

In Eq. (3.40a), the index pair (i,j) denotes the unique Gel'fand pattern having weight $e_i - e_j$, where $e_1 = (1\ 0\ 0), e_2 = (0\ 1\ 0), e_3 = (0\ 0\ 1)$. The invariant operators appearing in these relations have the following definitions:

$$\mathbf{I} = (3\mathbf{I}_2 - \mathbf{I}_1^2)/27 , \tag{3.41a}$$

with

$$\mathbf{I}_1 = \sum_i K_{ii} \ , \ \mathbf{I}_2 = \sum_{i,j} K_{ij} K_{ji} \ . \tag{3.41b}$$

The eigenvalues of $\mathbf{I}_1$ and $\mathbf{I}_2$ on $H_{[m]}$ are

$$I_1([m]) = m_{13} + m_{23} + m_{33} \ , \tag{3.42a}$$

$$I_2([m]) = \sum_{i=1}^{3} (m_{i3} + 3 - i)^2 - 2 \sum_{i=1}^{3} (m_{i3} + 3 - i) + 1 \ . \tag{3.42b}$$

We thus obtain the following relations on $H_{[m]}$:

$$K_{ij} = (-1)^j A([m]) E \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & & (i,j) & & \end{array} \right\rangle \ , \ i \neq j \ , \tag{3.43a}$$

$$K_{ii} = A_{ii}([m]) E \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle + B_{ii}([m]) E \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & 1 & & -1 & \\ & & 0 & & \end{array} \right\rangle$$

$$+ C_{ii}([m]) E \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle \ , \tag{3.43b}$$

where

$$\begin{aligned} A([m]) &= [(\dim[m]) I([m])/8]^{1/2} \ , \\ A_{11}([m]) = A_{22}([m]) &= A_{33}([m]) = [(\dim[m])/8]^{1/2} I_1([m])/3 \ , \\ -B_{11}([m]) = B_{22}([m]) &= A([m])/\sqrt{2} \ , \ B_{33}([m]) = 0 \ , \\ C_{11}([m]) = C_{22}([m]) &= -C_{33}([m])/2 = A([m])/\sqrt{6} \ . \end{aligned} \tag{3.43c}$$

These relations give explicitly the embedding of the symmetry group $U(3)$ in $U(\dim[m])$.

The commutators of the $U(3)$ generators K_{ij} with the general elements (3.38) of the Racah basis are given by

$$\left[K_{ij}, \ E \begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} \right] = \sum_{\alpha'} \left\langle \begin{matrix} [k] \\ \alpha' \end{matrix} \middle| K_{ij} \middle| \begin{matrix} [k] \\ \alpha \end{matrix} \right\rangle E \begin{pmatrix} \lambda \\ [k] \\ \alpha' \end{pmatrix} \ , \tag{3.44}$$

where $< \cdots |K_{ij}| \cdots >$ denotes the standard Gel'fand-Zetlin matrix elements of the $U(3)$ generators K_{ij}. These relations show that *the Racah basis of the Lie algebra of* $U(\dim[m])$ *consists of irreducible tensor operators with respect to* $U(3)$. Globally, this relation is expressed by

$$T_U E \begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} T_{U^{-1}} = \sum_{\alpha'} D^k_{\alpha'\alpha}(U) E \begin{pmatrix} \lambda \\ [k] \\ \alpha' \end{pmatrix} \text{ , each } U \in U(3) \,. \tag{3.45}$$

H. Structure Constants in the Racah Basis of the Lie Algebra of U(dim[m]):

We now apply relation (2.26) and use the product law of $U(3)$ unit tensor operators to obtain the following relations for the Racah basis of the Lie algebra of $U(\dim[m])$:

$$\left[E \begin{pmatrix} \lambda' \\ [k'] \\ \alpha' \end{pmatrix} , E \begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} \right] = \sum_{\begin{pmatrix} \lambda'' \\ [k''] \\ \alpha'' \end{pmatrix}} A \begin{bmatrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{bmatrix} E \begin{pmatrix} \lambda'' \\ k'' \\ \alpha'' \end{pmatrix} \text{ ,} \tag{3.46a}$$

where the *structure constants* in the relation are given by

$$\begin{aligned} A \begin{bmatrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{bmatrix} &= \left[\frac{\dim[k] \ \dim[k'] \ \dim[k'']}{\dim[m] \ \dim[m] \ \dim[m]} \right]^{1/2} \\ &\times \left(\begin{Bmatrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{Bmatrix} ([m]) - \begin{Bmatrix} \lambda'' & \lambda & \lambda' \\ [k''] & [k] & [k'] \\ \alpha'' & \alpha & \alpha \end{Bmatrix} ([m]) \right) \text{ ,} \end{aligned} \tag{3.46b}$$

with (see Eqs. (3.27)-(3.28))

$$\begin{aligned} \begin{Bmatrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{Bmatrix} ([m]) &= \sum_{\gamma'} \left\langle \begin{matrix} [k''] \\ \alpha'' \end{matrix} \middle| \left\langle \begin{matrix} \gamma' \\ [k'] \\ \alpha' \end{matrix} \right\rangle \middle| \begin{matrix} [k] \\ \alpha \end{matrix} \right\rangle \\ &\times \left\{ \begin{pmatrix} [k''] \\ \lambda'' \end{pmatrix} \begin{pmatrix} \gamma' \\ [k'] \\ \lambda' \end{pmatrix} \begin{pmatrix} [k] \\ \lambda \end{pmatrix} \right\} ([m]) \,. \end{aligned} \tag{3.46c}$$

This result for the structure constants is the $U(3)$ analogue of the $SU(2)$ result given by Eq. (2.28). It is to be noted, however, that the $U(\dim[m])$ structure constants given by Eq. (3.46b-c) entail a summation over the operator patterns γ'; that is, *the structure constants do not in the general case assume a factored form into a product of a WCG coefficient and a Racah coefficient.* This result complicates the situation for determining Lie algebraic zeroes of the $U(3)$ Racah coefficients, as we discuss in Section V.

I. Action of U(dim[m]) on $H_{[m]}$:

Let $V \in U(\dim[m])$, and let the elements of V be enumerated by the $U(2)$ Gel'fand patterns m' and m described in Eqs. (3.32)-(3.36). Then the action of $U(\dim[m])$ on $H_{[m]}$ is given by

$$S_V \left| \begin{matrix} [m] \\ m \end{matrix} \right\rangle = \sum_{m'} V_{m'm} \left| \begin{matrix} [m] \\ m' \end{matrix} \right\rangle , \text{ each } V \in U(\dim[m]) \,. \tag{3.47}$$

In particular, since $D^{[m]}(U) \in U(\dim[m])$, each $U \in U(3)$, we find that

$$S_{D^{[m]}(U)} = T_U \,. \tag{3.48}$$

This result shows that the space $H_{[m]}$ carries the fundamental representation

$$D^{[10\cdots 0]}(V) = V \tag{3.49}$$

of $U(\dim[m])$ and that this representation, when restricted to the symmetry group $U(3)$ as embedded in $U(\dim[m])$ in the Racah basis, reduces to the irreps of this $U(3)$:

$$D^{[10\cdots 0]}(D^{[m]}(U)) = D^{[m]}(U) \text{ , each } U \in U(3) \,. \tag{3.50}$$

IV. RACAH BASIS FOR THE LIE ALGEBRA OF ANY SUBGROUP G ⊂ U(dim[m]):

Let G be an arbitrary subgroup of $U(\dim[m])$:

$$G \subset U(\dim[m]) \,. \tag{4.1}$$

Let the Lie algebra of G have basis

$$\{X_1, X_2, \ldots, X_p\} \tag{4.2}$$

with commutation relations

$$[X_r, X_s] = \sum_t C^t_{rs} X_t \,, \quad r, s = 1, 2, \ldots, p \,, \tag{4.3}$$

where $\{C^t_{rs}\}$ denote the structure constants.

We may restrict the fundamental $[1\ 0\ \cdots\ 0]$ irrep of $U(\dim[m])$ to G and obtain a representation of G by matrices of dimension, $\dim[m]$. We denote the corresponding representation of the basis elements $\{X_r\}$ of the Lie algebra by

$$\{M_1, M_2, \ \dots\ , M_p\}\ . \tag{4.4}$$

These $(\dim[m]) \times (\dim[m])$ matrices then satisfy the commutation relations (4.3b):

$$[M_r, M_s] = \sum_t C^t_{rs}\, M_t\ , \quad r, s = 1, 2, \ \dots\ , p\ . \tag{4.5}$$

We next obtain the realization of the Lie algebra of G on the vector space $H_{[m]}$ in the following way: Define the linear map

$$L_r : H_{[m]} \rightarrow H_{[m]} \tag{4.6a}$$

by

$$L_r = \sum_{m', m} (M_r)_{m'm}\, E_{m'm}\ , \tag{4.6b}$$

where $\{E_{m'm}\}$ is the set of basis elements of the $U(\dim[m])$ Lie algebra, as described by Eqs. (3.32)-(3.36). The maps in the set

$$\{L_1, L_2, \ \dots\ , L_p\} \tag{4.7}$$

then also satisfy the commutation relations:

$$[L_r, L_s] = \sum_t C^t_{rs}\, L_t\ , \quad r, s = 1, 2, \ \dots\ , p\ . \tag{4.8}$$

The operators L_r in the set (4.6) can be expressed in terms of the Racah basis of the Lie algebra of $U(\dim[m])$ by using Eq. (3.39), which relates the Weyl and Racah basis:

$$L_r = \sum_{\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix}} R_r \begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} E \begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix}\ , \tag{4.9a}$$

where the coefficients $R_r(\ \cdot\ )$ are defined by

$$R_r \begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} = \sum_{m', m} (M_r)_{m'm}\, R \left[\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} \begin{pmatrix} m \\ [m] \\ m' \end{pmatrix} \right]\ , r = 1, 2, \ \dots\ , p\ . \tag{4.9b}$$

Using identity (4.9a) for the L_r in the commutation relations (4.7), and Eq. (3.46a) for the commutator of Racah basis elements, we obtain the following relation between structure constants:

$$\sum_{\begin{pmatrix} \lambda' \\ [k'] \\ \alpha' \end{pmatrix}\begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix}} R_r \begin{pmatrix} \lambda' \\ [k'] \\ \alpha' \end{pmatrix} R_s \begin{pmatrix} \lambda \\ [k] \\ \alpha \end{pmatrix} A \begin{bmatrix} \lambda'' & \lambda' & \lambda \\ [k''] & [k'] & [k] \\ \alpha'' & \alpha' & \alpha \end{bmatrix}$$

$$= \sum_t C^t_{rs} R_t \begin{pmatrix} \lambda'' \\ [k''] \\ \alpha'' \end{pmatrix} . \tag{4.10}$$

An important point to note here is that, once the matrix realization

$$X_r \to M_r \, , \quad r = 1, 2, \ldots, p \tag{4.11}$$

of the Lie algebra of G is given, the operator realization of this algebra on the space $H_{[m]}$,

$$X_r \to L_r \, , \quad r = 1, 2 \ldots, p \, , \tag{4.12}$$

in the Racah basis, as given by Eqs. (4.9), is *uniquely* determined.

V. ZEROES OF U(3) RACAH COEFFICIENTS

We have shown in the Section III how the $U(3)$ Racah coefficients enter into the structure constants of the Lie algebra of $U(\dim[m])$ in the Racah basis of this algebra. The symbol for these coefficients and some of its properties are given in Eqs. (3.26)-(3.31). The coefficient is zero, by definition, whenever the integers entering into the arrays occurring in the symbol fall outside their domains as described by the generalized triangle conditions in (i)-(v) given in Eq. (3.29b).

It is a well-known fact that the $SU(2)$ Racah coefficients possess infinite families of zeroes, even when all the triangle conditions are fulfilled and all symmetries taken into account. A few of these zeroes (nine, in fact, as of 1988) have found explanation in terms of a group G lying between $SU(2)$ [or $SO(3)$] and $U(2j+1)$:

$$SU(2) \subset G \subset U(2j+1) \, , \text{ for some } j \, ,$$

or

$$SO(3) \subset G \subset U(2j+1) \, , \text{ for some } j \, . \tag{5.1}$$

(See Ref. 11 for summaries as of 1986; see also Refs. 10,41-46); since the Lie algebra of G is a sub-Lie algebra of that of $U(2j+1)$, containing also the $SU(2)$ [or $SO(3)$] Lie algebra, the Lie algebra of G can be realized in terms of the Racah basis. This then implies relations on the structure constants given by Eq. (2.28), leading in some instances to a required vanishing of a Racah coefficient. We call zeroes of this type ***Lie algebraic zeroes.***

Infinite families of zeroes of the Racah coefficients, including the Lie algebraic ones, have also been obtained as the solutions of classical Diophantine equations that arise directly from the explicit form of these coefficients. Brudno[47] initiated these Diophantine equation studies, and, since, a number of publications[48-57] along these lines have followed. (No comprehensive theory of all zeroes of $SU(2)$ Racah coefficients has yet appeared, to our knowledge.) (Aside from the discovery of several Lie algebraic zeroes, the occurrence of numerous zeroes was first shown numerically in Ref. 58; see also Ref. 59.)

The question of zeroes of the $SU(3)$ Racah coefficients is complicated beyond that of $SU(2)$ by another structural feature: The structure constants in the Racah basis given by Eq. (3.46a) no longer appear in a factored form as they do in the $SU(2)$ case of Eq. (2.28). This situation is unavoidable, since it is a direct consequence of the multiplicity structure of the Kronecker product reduction. On the other hand, if G is a group such that

$$SU(3) \subset G \subset U(\dim m)\,, \tag{5.2}$$

then the Racah basis of the Lie algebra of G must close with attendant relations between the structure constants. Does this imply also some $SU(3)$ Racah coefficient zeroes? We examine this question more closely below for the exceptional Lie group E_6, which has the embedding

$$SU(3) \subset E_6 \subset SU(27)\,. \tag{5.3}$$

Still another feature enters into the discussion of $U(3)$ Racah coefficient vanishings. Certain zeroes are implied by the canonical splitting conditions that resolve the $U(3)$ multiplicity. These zeroes are, in fact, a consequence of the fact that a certain class of Racah invariants must be the null invariant operator 0 in consequence of null space structural properties of operator patterns (see Ref. 34). These zeroes go beyond the triangle rules of Eq. (3.29b). It is proved in Ref. 34 (see Eqs. (3.15) of this reference) that these structural zeroes include the following ones among the coefficients described in Eqs. (3.28) (3.29):

$$\left\{ \begin{pmatrix} [k''] \\ \lambda'' \end{pmatrix} \begin{pmatrix} \gamma' \\ [k'] \\ \lambda' \end{pmatrix} \begin{pmatrix} [k] \\ \lambda \end{pmatrix} \right\} \quad 0\,, \tag{5.4}$$

whenever the following condition is fulfilled:

$$i + i' - i'' \geq [k_1' - \Delta_1(\gamma')] - [k_3' - \Delta_3(\gamma')] + 2\,, \tag{5.5}$$

where the indices i, i', i'' have the following definitions in terms of the operator patterns appearing in the symbol (3.29a):

$$i = \begin{cases} k_1 - \ell + 1\,, & \text{for } k_2 \geq 0 \\ -k_3 - \ell + 1\,, & \text{for } k_2 < 0 \end{cases}\;; \tag{5.6a}$$

$$i' = \begin{cases} k_1' - \ell' + 1\,, & \text{for } k_2' \geq 0 \\ -k_3' - \ell' + 1\,, & \text{for } k_2' < 0 \end{cases}\;; \tag{5.6b}$$

$$i'' = \begin{cases} k_1'' - \ell'' + 1\,, & \text{for } k_2'' \geq 0 \\ -k_3'' - \ell'' + 1\,, & \text{for } k_2'' < 0 \end{cases}\;. \tag{5.6c}$$

The indices i, i', i'' arise in the canonical labelling of the operator patterns in a given multiplicity set in the following way (see Refs. 32-34). It is sufficient to illustrate the rule for the index i, since the others are obtained similarly. $U(3)$ operator patterns may be ordered by two different rules, which, it turns out, are compatible. In the first rule, we write for $k_2 \geq 0$:

$$\begin{pmatrix} k_1 & k_2 & k_3 \\ & \Gamma_{k_1-\ell+1} & \end{pmatrix} = \begin{pmatrix} k_1 & & k_2 & & k_3 \\ & \ell & & -\ell & \\ & & 0 & & \end{pmatrix}\;; \tag{5.7a}$$

that is,

$$\Gamma_{k_1-\ell+1} = \lambda = \begin{pmatrix} \ell & & -\ell \\ & 0 & \end{pmatrix}\,, \quad \ell = k_1, k_1 - 1, \ldots, k_2\,. \tag{5.7b}$$

We now apply the order rule on such patterns given by Eqs. (3.35) to obtain

$$\Gamma_1 > \Gamma_2 > \cdots > \Gamma_M\,, \quad M = k_1 - k_2 + 1\,. \tag{5.7c}$$

Similarly, for $k_2 < 0$, we write

$$\Gamma_{-k_3-\ell+1} = \lambda = \begin{pmatrix} \ell & & -\ell \\ & 0 & \end{pmatrix}\,, \quad \ell = -k_2, -k_2 + 1, \ldots, -k_3\,, \tag{5.8a}$$

with again

$$\Gamma_1 > \Gamma_2 > \cdots > \Gamma_M\,, \quad M = k_2 - k_3 + 1\,. \tag{5.8b}$$

These inequalities on operator patterns then accord exactly with the nested properties of the null space of the unit tensor operators

$$\left\langle \begin{matrix} & \Gamma_t & \\ k_1 & k_2 & k_3 \\ & \bullet & \end{matrix} \right\rangle , \quad t = 1,2, \ldots , M , \tag{5.9a}$$

as given by

$$N(\Gamma_1) \supset N(\Gamma_2) \supset \cdots \supset N(\Gamma_M) . \tag{5.9b}$$

Operator patterns were, of course, introduced precisely to accommodate this nested null space structure of the associated unit tensor operators.

The results given in Eqs. (5.4)-(5.9) allows us to identify ***structural zeroes*** of the Racah coefficient associated with the canonical resolution of the multiplicity. We use this result below

Let us return now to the problem, for Racah coefficients, originating from the existence of the group-subgroup chain (5.3). Our first problem is to obtain the generators of E_6 in terms of the Racah basis of $SU(27)$. The 27-dimensional irrep of E_6 is complex,[60] which implies that the generators of this representation are antihermitian. Thus, for the E_6 Lie algebra, we may take the basis elements

$$\{M_1, M_2, \ldots , M_p\} , \quad \text{with } p = 78 , \tag{5.10a}$$

of Eq. (4.4) to be antihermitian

$$M_r^\dagger = -M_r , \quad r = 1,2, \ldots , 78 . \tag{5.10b}$$

Using this result in Eq. (4.6b), we obtain

$$L_r^\dagger = -L_r , \quad r = 1, \ldots , 78 , \text{ on } H_{[4\ 2\ 0]} . \tag{5.11}$$

Thus, the Lie algebra of E_6 is realized on the space $H_{[4\ 2\ 0]}$ as antihermitian operators. For the consistency of Eq. (5.3), let us note that the $SU(3)$ Lie algebra can also be realized by 27×27 real, skew symmetric matrices. This representation is equivalent to the [4 2 0] irrep, and can be obtained by using the Gell Mann[61] generators of $SU(3)$.

We thus obtain eight of the E_6 generators as the eight $SU(3)$ generators, as given in the Racah basis by Eqs. (3.40) (3.43). The remaining seventy must come in conjugate pairs to realize the conjugation property (5.11). All maps $H_{[4\ 2\ 0]} \to H_{[4\ 2\ 0]}$ are labelled by the irreps occurring in the reduction of the Kronecker product $[0\ \ -2\ \ -4] \times [4\ 2\ 0]$. This reduction is given by Eqs. (3.15)-(3.17) to be

$$
\begin{aligned}
&[0\ \ -2\ \ -4] \times [4\ \ 2\ \ 0] = \\[1ex]
&\underset{125}{[4\ \ 0\ \ -4]} \oplus \underset{81}{[4\ \ -1\ \ -3]} \oplus \underset{81}{[3\ \ 1\ \ -4]} \\[1ex]
&\oplus \underset{28}{[4\ \ -2\ \ -2]} \oplus \underset{28}{[2\ \ 2\ \ -4]} \oplus 2\underset{64}{[3\ \ 0\ \ -3]} \oplus 2\underset{35}{[3\ \ -1\ \ -2]} \oplus 2\underset{35}{[2\ \ 1\ \ -3]} \\[1ex]
&\oplus 3\underset{27}{[2\ \ 0\ \ -2]} \oplus \underset{10}{[2\ \ -1\ \ -1]} \oplus \underset{10}{[1\ \ 1\ \ -2]} \oplus 2\underset{8}{[1\ \ 0\ \ -1]} + \underset{1}{[0\ \ 0\ \ 0]}\ ,
\end{aligned}
\tag{5.12}
$$

where we have written the dimension below each irrep label. We see from this result that the only irrep labels that qualify for labelling the remaining generators of E_8 in the Racah basis are the thirty-five dimensional conjugate irreps

$$[2\ \ 1\ \ -3] \text{ and } [3\ \ -1\ \ -2]\ . \tag{5.13}$$

The explicit conjugation relation for the corresponding unit tensor operators is

$$
\left\langle \begin{array}{ccc} & \lambda & \\ 2 & 1 & -3 \\ & \alpha & \end{array} \right\rangle^{\dagger} = (-1)^{\phi(\alpha)} \left\langle \begin{array}{ccc} & \lambda & \\ 3 & -1 & -2 \\ & \bar{\alpha} & \end{array} \right\rangle ,
$$

$$
\lambda = \begin{pmatrix} & 0 & \\ 1 & & -1 \end{pmatrix} \text{ or } \begin{pmatrix} & 0 & \\ 2 & & -2 \end{pmatrix} , \tag{5.14a}
$$

where

$$
\phi(\alpha) = \alpha_{11} + \alpha_{12} + \alpha_{22}\ , \ \bar{\alpha} = \begin{pmatrix} -\alpha_{22} & & -\alpha_{12} \\ & -\alpha_{11} & \end{pmatrix} . \tag{5.14b}
$$

We conclude from these relations that the orthogonal matrix defined by Eq. (3.37a) has the property

$$
\begin{aligned}
&R\left[\begin{pmatrix} & \lambda & \\ 2 & 1 & -3 \\ & \alpha & \end{pmatrix}\begin{pmatrix} m \\ [m] \\ m' \end{pmatrix}\right] \\
&\qquad = (-1)^{\phi(\alpha)}\, R\left[\begin{pmatrix} & \lambda & \\ 3 & -1 & -2 \\ & \alpha & \end{pmatrix}\begin{pmatrix} m' \\ [m] \\ m \end{pmatrix}\right]
\end{aligned}
\tag{5.15a}
$$

Using this result in Eq. (4.9b), together with the antihermitian property of the matrix M_r, now gives

$$R_r^*\begin{pmatrix} & \lambda & \\ 2 & 1 & -3 \\ & \alpha & \end{pmatrix} = -(-1)^{\phi(\alpha)}\, R_r\begin{pmatrix} & \overline{\lambda} & \\ 3 & -1 & -2 \\ & \overline{\alpha} & \end{pmatrix} \tag{5.15b}$$

for each of the operator patterns λ in Eq. (5.14a) (note that $\overline{\lambda} = \lambda$).

We now use relation (5.15a) in Eq. (4.9a), selecting only the terms $[2\ \ 1\ \ -3]$ and $[3\ \ -1\ \ -2]$ from the sum, to conclude that the remaining generators of E_6 in the Racah basis are given by

$$\begin{aligned} L_r = {} & \sum_{\ell=1}^{2}\sum_{\alpha} R_r\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 2 & & 1 & & -3 \\ & & \alpha & & \end{pmatrix} E\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 2 & & 1 & & -3 \\ & & \alpha & & \end{pmatrix} \\ & - \sum_{\ell=1}^{2}\sum_{\alpha} R_r^*\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 2 & & 1 & & -3 \\ & & \alpha & & \end{pmatrix} (-1)^{\phi(\alpha)}\, E\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 3 & & -1 & & -2 \\ & & \overline{\alpha} & & \end{pmatrix}, \end{aligned} \tag{5.16a}$$

where $r = 9, 10, \ldots, 78$, since we can choose the generators

$$\{L_1, L_2, \ldots, L_8\} \tag{5.16b}$$

to be those of the $SU(3)$ subalgebra. This set of operators then satisfies the antihermitian property:

$$L_r^{\dagger} = -L_r\,, \quad r = 1, 2, \ldots, 78\,. \tag{5.17}$$

We observe again that once the 27-dimensional antihermitian generators (5.10a) of the 27-dimensional representation of E_6 are specified, there is no freedom left in expressing the generators L_r in terms of the Racah basis, since the R_r coefficients in Eqs. (5.16a,b) are uniquely obtained from Eq. (4.9b).

It follows from Eq. (5.16a) that the following operators are a basis for these generators:

$$L_\alpha^{(1)} = \sum_{\ell=1}^{2}\left[E\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 2 & & 1 & & -3 \\ & & \alpha & & \end{pmatrix} - (-1)^{\phi(\alpha)}\, E\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 3 & & -1 & & -2 \\ & & \overline{\alpha} & & \end{pmatrix}\right], \tag{5.18a}$$

$$L_\alpha^{(2)} = i\sum_{\ell=1}^{2}\left[E\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 2 & & 1 & & -3 \\ & & \alpha & & \end{pmatrix} + (-1)^{\phi(\alpha)}\, E\begin{pmatrix} & & 0 & & \\ & \ell & & -\ell & \\ 3 & & -1 & & -2 \\ & & \overline{\alpha} & & \end{pmatrix}\right]. \tag{5.18b}$$

where α runs over all thirty-five Gel'fand patterns for the given irrep labels. These operators then satisfy the following identities on the space $H_{[4\ 2\ 0]}$:

$$L_\alpha^{(1)\dagger} = -L_\alpha^{(1)}\,, \quad L_\alpha^{(2)\dagger} = -L_\alpha^{(2)}\,. \tag{5.19}$$

Let us now consider the consequences of the E_6 embedding (5.3). We have, first of all, that the commutators

$$\left[E\begin{pmatrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & & (i,j) & & \end{pmatrix},\ L_\alpha^{(a)} \right]\,, \quad a = 1,2\,, \quad \begin{array}{l} i \neq j = 1,2,3 \\ i = j = 1,2\,, \end{array} \tag{5.20}$$

must close on the generators of E_6. These commutators must, by construction of the Racah basis, close automatically on the generators of E_6 because of the irreducible tensor property expressed by Eq. (3.44) (see Eqs. (3.40) and (3.43)). That is, the closure of the commutators (5.20) is a property of the Racah basis, and not of the E_6 embedding. Indeed, in consequence of the irreducible tensor operator property (3.44) and the generator relations (3.40), the structure constants for the Racah basis must obey the relations:

$$A\begin{bmatrix} \lambda'' & & & 0 & & & \lambda \\ [k''] & & 0 & & 0 & & [k] \\ \alpha'' & [1 & & 0 & & -1] & \alpha \\ & & & (i,j) & & & \end{bmatrix} = \delta_{[k''][k]} \left\langle \begin{matrix}[k] \\ \alpha''\end{matrix} \,\middle|\, \left\langle \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & -1 \\ & & (i,j) & & \end{matrix} \right\rangle \,\middle|\, \begin{matrix}[k] \\ \alpha\end{matrix} \right\rangle\,,$$

$$\text{for } i \neq j = 1,2,3;\ i = j = 1,2\,; \tag{5.21a}$$

$$A\begin{bmatrix} & & 0 & & \\ \lambda'' & & 0\ \ 0 & & \lambda \\ [k''] & [0 & 0 & 0] & [k] \\ \alpha'' & & 0\ \ 0 & & \alpha \\ & & 0 & & \end{bmatrix} = \delta_{[k''][k]}\, \delta_{\lambda''\lambda}\, \delta_{\alpha''\alpha}\,. \tag{5.21b}$$

These relations already lead to interesting and nontrivial identities between WCs and Racah coefficients when the structure constants given by Eqs. (3.46b,c) are substituted for the left hand side in Eq. (5.21a). These relations are, as pointed out above, not properties of the E_6 embedding, but rather of the canonical splitting and null space conditions defining the solution of the multiplicity problem.

For the possibility of a $SU(3)$ Racah coefficient vanishing in consequence of the E_6 embedding (5.3), one requires the closure of the commutators

$$[L_\alpha^{(a)}\,,\ L_{\alpha'}^{(a')}]\,, \quad a, a' = 1,2\,, \tag{5.22}$$

on the E_6 generators. In the next step, we substitute relations (5.18) for the generators into the commutator (5.20), using relation (3.46a) to obtain a linear combination of the Racah basis elements of the form

$$E\begin{pmatrix} \lambda'' \\ [k''] \\ \alpha'' \end{pmatrix} \text{ with } \begin{array}{l} [k''] \in [2 \quad 1 \; -3] \times [2 \quad 1 \; -3]\,, \\ [k''] \in [2 \quad 1 \; -3] \times [3 \; -1 \; -2]\,, \\ [k''] \in [3 \; -1 \; -2] \times [3 \; -1 \; -2]\,. \end{array} \tag{5.23}$$

For closure of the Lie algebra of E_6, the coefficients of each operator (5.23) must vanish for $[k''] \neq [1 \quad 0 \; -1], [2 \quad 1 \; -3], [3 \; -1 \; -2]$. This leads to linear relations between the structure constants.

To determine if there are any $SU(3)$ Racah coefficient zeroes associated with a linear relation between structure constant described above, one expresses these coefficients in terms of WCG and Racah coefficients using Eqs. (3.46b,c). One must at the same time account for the null space zeroes as described by Eqs. (5.5)-(5.9). The result of this is a set of relations between WCG and Racah coefficients. Finally, one must take into account the symmetry relations for WCG coefficients, as well as those for the Racah coefficients. Since such symmetry relations are not, at this time, known generally for generic irrep labels, it is a sizeable task to implement this procedure. It may, indeed, require numerical calculation of the relevant coefficients.

We have formulated here the rather intricate structural relations that underly the problem of zeroes of $SU(3)$ Racah coefficients. We hope to implement this process for particular coefficients in a future paper, using either more detailed knowledge of the required symmetries than presently available, or with the help of numerical calculations.

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